Pergamon

J. Appl. Maths Mechs, Vol. 61, No. 5, pp. 775–786, 1997

0021–8928/97 $24.00+0.00

PII: S0021-8928(97)00101-9

HIGHER APPROXIMATIONS OF THE TRANSONIC EXPANSION IN PROBLEMS OF UNSTEADY TRANSONIC FLOW†

A. N. BOGDANOV

Moscow

(*Received* 9 *August* 1994)

Modifications of the equation of unsteady transonic flow are investigated, which avoid the drawbacks which arise when using the transonic approximation (the Lin–Reissner–Tsien equations) and other equations of unsteady transonic flow to describe transonic flows. Several problems concerning flows of this kind (the stability of transonic flows, the formation of unsteady shock waves and the mathematical modelling of flows) are solved. © 1998 Elsevier Science Ltd. All rights reserved.

In the past, attempts to simplify the equations for the gas motion at sonic velocities have involved the use of a so-called transonic expansion in terms of a small deviation of the actual Mach number of the flow from unity. This is the simplification for unsteady transonic motion made by Lin *et al.* in [1]. Although the resulting equation (the LRT equation) retains many important features of transonic flows (such as non-linearity and spatial non-uniformity, and also describes regions of subsonic and supersonic flow), it has been shown to have serious defects. It gives infinite rates of downstream propagation of weak unsteady perturbations, and at every instant of time the wave fronts of perturbations arising from a point source are unclosed curves (parabolas). The Cauchy problem for the LRT equation is ill-posed, it does not describe high-frequency unsteady perturbations, etc. Nevertheless, as it is the simplest available, the LRT equation has been widely used to analyse unsteady transonic flows. A review of this work and some results can be found in [2, 3].

An investigation has been made of the correct formulation of boundary-value problems for the LRT equation.‡ Since its characteristics are the surfaces t = const, not all the initial data can be assigned in the plane $t = 0$. The condition imposed on the required function at $t = 0$ must be supplemented by conditions on the function and its first derivative with respect to the space coordinate x, given at $x = 0$. Furthermore, this derivative must be positive.

Several different attempts have been made to overcome the defects in the description of transonic flows based on the use of the LRT equation, mainly by adding various terms [4]. Research has shown that adding a term with a second time derivative to the LRT equation [5, 6] removes many of the difficulties mentioned above. The modified equation describes perturbations of any frequency (including high-frequency perturbations) propagating in any direction (including downstream). The Cauchy problem for this equation is well-posed, the cone of the dependence of the solution on the initial data can be constructed, an energy integral can be obtained as well as estimates of the solution for the initial and boundary-value problems.

Below we consider some problems in the theory of unsteady transonic flows based on the modified equation of unsteady transonic flows [6], as well as other modified equations which model flow of this kind.

In real conditions of boundary layer turbulence, there is a source of unsteady perturbations at the flow boundaries, and these spread in every direction in the flow field. Apart from those which are moving against the basic flow, progressive waves disappear rapidly, owing to the relatively high velocity with which they propagate. The perturbations which move upstream accumulate in regions where the supersonic flow decelerates to subsonic velocities. If there is no dissipation of energy of these perturbations, there is enough time for them to grow and, ultimately, destroy the smooth steady flow. It is probable [7] that the flow either becomes unsteady or contains shock waves, or both. This view is confirmed by experiment: there are stable plane transonic flows which contain steady shock waves [8]. Furthermore, a study of the behaviour of weak steady perturbations generated by irregularities of the flow boundaries

†*Prikl. Mat. Mekh.* Vol. 61, No. 5, pp. 798–811, 1997.

‡MAMONTOV, Ye. V., A contribution to the theory of unsteady transonic gas flows. Candidate dissertation 01.01.02. Lavrent'yev Institute of Hydrodynamics, Novosibirsk, 1973.

has shown that their growth can lead to the formation of shock waves inside the supersonic flow regions [9, 10]. The shock waves can arise from the growth of unsteady perturbations which arrive from outside in the transonic flow region.

Serious difficulties are encountered when investigating the stability of transonic flows with respect to small unsteady perturbations, owing to the fact that the equations of transonic approximation are non-linear [3]. Also, these partial differential equations for a steady background (the stability of which is being investigated) are of a varying elliptic-hyperbolic type. For this reason, the stability of transonic flow has only been determined using a quasi-one-dimensional approximation [11–13] and linearization of the LRT equation [14] (which, as we have mentioned, has a number of drawbacks). The various attempts to perform a more general analysis have proved unsatisfactory, even to their authors [7].

The issue on the appearance of shock waves in transonic flows has previously been considered in the steady case only. The examples constructed of flows with steady shock waves are realized in Laval nozzles with an extended transition section [2].

The use of the transonic approximation in its classical form [3] distorts the structure of the region in which a solution is sought: a family of degenerate characteristics (the trivial planes t = const) arises, and the cone of the dependence is destroyed and degenerates into a paraboloid of influence. Nor has it been possible to obtain the energy integral normally used in the theory of partial differential equations to obtain estimates for the solution of equations of this kind. Note that an energy integral can be constructed if a mixed problem is posed for the LRT equation with data in the planes $t = 0$ and $x = 0$ (see the reference in the footnote).

Below, using a modified equation of unsteady transonic flow [6], by means of which it is possible to describe perturbations which propagate downstream, we analyse the development of small unsteady perturbations in transonic flow. We consider the issue on the formation of the envelope of the acoustic characteristics of one family, implying that an unsteady shock wave has arisen. We show that the modified equation does not describe the flow dynamics in a direction transverse to the main flow. Moreover, we obtain an energy integral for unsteady transonic flow. For mixed problems the results largely depend on the boundary on which the boundary conditions are specified, since (like the LRT equation) the proposed modified equation is not invariant when the axes of the space coordinates are interchanged.

Since the modified LRT equation does not permit a sufficiently accurate solution of the theoretical problem of transonic flows (the results are sometimes simply false), by an extension of earlier studies on a modification of the equation of unsteady transonic flow, in Section 5 we give an equation which can be used to study the non-linear evolution of wave fronts propagating at an angle to the direction of the main flow. This new modification involves taking account of terms of the next order of smallness in the usual transonic expansion. We solve the problems described in Sections 1–4 using this equation.

The proposed modified equation of unsteady transonic flow enables us to obtain a mathematical model of the operation of a transonic nozzle under unsteady conditions [15].

Note that there have been numerical calculations of unsteady transonic flows using equations other than the LRT equation for the model. Thus, the modified equation in [16], which contains several of the terms proposed in the equation below, has been used to give a better description of the flow, but without a specific proof of the modification or any comment thereon.

1. WEAK UNSTEADY PERTURBATIONS OF TRANSONIC FLOW AND THE MODIFIED LIN–REISSNER–TSIEN EQUATION

The investigation below concerns unsteady transonic flow in a channel (nozzle) with plane geometry. The channel axis is taken to coincide with the X axis.

In the given approximation, which involves an expansion of the flow parameters in terms of the small derivation of the actual Mach number from unity (the conventional transonic expansion), the shock waves (if there are any) remain weak, the vorticity arising in the flow is low, and therefore we can introduce the velocity potential Φ: $u = \Phi_X$, $v = \Phi_Y$. The equation for the potential Φ follows from the equations of motion and continuity

$$(a^2 - \Phi_X^2)\Phi_{XX} - 2\Phi_X\Phi_Y\Phi_{XY} + (a^2 - \Phi_Y^2)\Phi_{YY} - 2\Phi_X\Phi_{XT} - 2\Phi_Y\Phi_{YT} - \Phi_{TT} = 0 \tag{1.1}$$

the velocity of sound a being determined by the equation of the conservation of energy

$$\Phi_T + \frac{1}{2}(\Phi_X^2 + \Phi_Y^2) + \frac{a^2}{\gamma - 1} = \frac{U^2}{2} + \frac{1}{\gamma - 1}a_\infty^2 \tag{1.2}$$

a_∞ and U are the characteristic velocity of sound and the velocity of main flow in the channel and γ is the adiabatic exponent.

We will introduce new dimensionless space coordinates and time

$$Lx = X, \quad Ly = \delta^{1/3} Y, \quad t = \delta^{2/3} U L^{-1} T$$

(L is the characteristic length and $\delta \ll 1$) and an expansion of the potential in the form [3]

$$\Phi = U(Lx + \delta^{2/3} L\varphi(x, y, t) + \ldots)$$

(transonic flow is represented as a perturbation of one-dimensional flow along the x axis with constant velocity U). We have

$$\Phi_X = U(1 + \delta^{2/3}\varphi_x), \quad \Phi_Y = U\delta\varphi_y, \quad \Phi_T = U^2\delta^{2/3}\varphi_t \tag{1.3}$$

and the system of equations (1.1) and (1.2) becomes the equation

$$\left\{\delta^{-2/3}\left(\frac{a_\infty^2}{U^2} - 1\right) - \delta^{2/3}(\gamma - 1)\varphi_t - (\gamma+1)\varphi_x - \frac{\gamma+1}{2}\delta^{2/3}\varphi_x^2 - \frac{\gamma-1}{2}\delta^{4/3}\varphi_y^2\right\}\varphi_{xx} -$$
$$-2(1+\delta^{2/3}\varphi_x)\delta^{2/3}\varphi_y\varphi_{xy} + \left(\frac{a_\infty^2}{U^2} - (\gamma-1)\delta^{2/3}(\varphi_x + \delta^{2/3}\varphi_t) - \frac{\gamma-1}{2}\delta^{4/3}\varphi_x^2 - \frac{\gamma+1}{2}\delta^2\varphi_y^2\right)\varphi_{yy} -$$
$$-2(1+\delta^{2/3}\varphi_x)\varphi_{xt} - 2\delta^{4/3}\varphi_y\varphi_{ty} - \delta^{2/3}\varphi_{tt} = 0 \tag{1.4}$$

For terms of order $\delta^{4/3}$ we obtain [3]

$$(K - (\gamma+1)\varphi_x \quad \varphi_{xx} + \varphi_{yy} - 2\varphi_{xt} = 0, \quad K = \bar{\delta}^{2/3}(1 - M_\infty^2), \quad M_\infty = U / a_\infty \tag{1.5}$$

Equation (1.5) is known as the LRT equation. It does not hold for high frequencies of perturbations $\Phi_{TT} > \Phi_T$.

According to the estimates (1.3)

$$\Phi_{TT} = L^{-1}U^3\delta^2\varphi_{tt} \tag{1.6}$$

Assuming that the quantity φ_{tt} can be very large, we retain the term (1.6) when writing the transonic limit. We obtain ($\varepsilon = \delta^{2/3} \ll 1$)

$$(K - (\gamma+1)\varphi_x)\varphi_{xx} + \varphi_{yy} - 2\varphi_{xt} - \varepsilon\varphi_{tt} = 0 \tag{1.7}$$

The coefficient of Φ_{xx} is an alternating-sign quantity, depending on whether the flow is sub- or supersonic, since

$$a^2 - u^2 = (K - (\gamma+1)\varphi_x)\delta^{2/3} + O(\delta^{4/3}).$$

Whatever the sign of that coefficient, Eq. (1.7) is hyperbolic, and since the coefficient depends on the first derivative of the potential, it is quasi-linear.

Characteristic surfaces can be obtained for Eq. (1.7) (or (1.5)) [3]. The characteristic surfaces of Eq. (1.7) are the cones

$$K^*t^2 + 2t(x - x_0) - \varepsilon(x - x_0)^2 = (y - y_0)^2(1 + \varepsilon K^*) \tag{1.8}$$

If $K^* = K - (\gamma+1)\varphi_x$ = const, for each fixed t, Eq. (1.8) gives an ellipse in the x, y plane (Fig. 1).

The case $\varepsilon = 0$ has already been explored [3]. In that case (supposing also that $x_0 = y_0 = 0$) Eq. (1.8) transforms to an equation of the form

$$K^*t^2 + 2tx = y^2 \tag{1.9}$$

The solutions of (1.9) for given t are parabolas in the x, y plane. These solutions imply infinite velocities

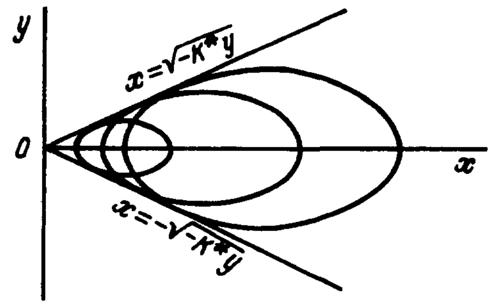

Fig. 1.

of downstream propagation of the wave fronts of weak perturbations and are unrealistic from a physical standpoint. Note that when $\varepsilon = 0$, the planes t = const will also be characteristic surfaces. Infinite velocities of downstream propagation of perturbations and the ill-posedness of the Cauchy problem for the LRT equation are a formal consequence of this.

We will consider the envelope of wave fronts (1.9) as t changes. This envelope (for K^* = const) is defined by the system

$$K^*t^2 + 2tx = y^2, \quad K^*t + x = 0$$

from which we obtain (when $K^* < 0$ the flow is supersonic)

$$x = \pm\sqrt{-K^*}\, y \tag{1.10}$$

The envelope (1.10) determines the Mach angle. It can be verified that the wave fronts (1.8) issuing from the perturbation source at the point $x_0 = y_0 = 0$ have the same envelope. This suggests that the accuracy of the description of perturbations propagating upstream is adequate.

There have been other modifications of the equation of unsteady transonic flow. It was proposed in [5] that account should be taken of the unsteady non-linear component (the term of order ε according to Eq. (1.4)). Then instead of (1.7) we have the equation

$$(K - (\gamma+1)\varphi_x - \varepsilon(\gamma-1)\varphi_t)\varphi_{xx} + \varphi_{yy} - 2\varphi_{xt} - \varepsilon\varphi_{tt} = 0 \tag{1.11}$$

used in [5] to analyse transonic unsteady gas flows in a channel. Some features of the analysis of transonic flows revealed Eq. (1.11), and other modified equations will be considered below.

Equation (1.4), on the basis of which the approximate transonic equations are derived, is not invariant under an interchange of the axes of the space coordinates. Moreover, when a rough approximation is used, any weaker non-linearity of the equation in the direction of the transverse main flow is simply ignored. This can lead to incomplete and incorrect results and conclusions on the properties of transonic flows when studied in a spatial formulation.

2. THE STABILITY OF TRANSONIC FLOWS

The main conclusion of all previous investigations of the stability of transonic flows is that accelerated flows which pass through the speed of sound are stable, and decelerated flows are unstable [7, 8, 11, 14, 17].

The view has been expressed [7] that the behaviour of perturbations at transonic points is typically non-linear, and it is normally impossible to use linearization when analysing the stability.

We will show that the generally accepted transonic approximation, although it does preserve non-linearity, does not take any account of perturbations propagating downstream, while the modification of the LRT equation proposed in [5] gives incorrect results for perturbations which propagate transverse to the direction of the main flow. The influence of non-linearity will be taken into account below in Section 3.

We will consider the behaviour of weak unsteady perturbations

$$\varphi = \varphi^0(x, y) + \tilde{\delta}\varphi(x, y, t), \quad \tilde{\delta} \ll 1$$

Linearizing Eq. (1.7) with respect to $\tilde{\delta}$, we have

$$K^{*0}\tilde{\delta}\varphi_{xx} - (\gamma+1)\tilde{\delta}\varphi_x\varphi^0_{xx} + \tilde{\delta}\varphi_{yy} - 2\tilde{\delta}\varphi_{xt} - \varepsilon\tilde{\delta}\varphi_{tt} = 0 \quad (K^{*0} = K - (\gamma+1)\varphi^0_x) \tag{2.1}$$

We let the perturbations have a plane front and consider a local analysis of their behaviour. We will assume that the coefficients of Eq. (2.1) are constant and we will seek a solution in the form

$$(\tilde{\delta}\varphi \sim \exp(i(-\omega t + kx + ly)) \tag{2.2}$$

Substituting (2.2) into (2.1), we obtain the dispersion relation

$$-K^{*0}k^2 - ik(\gamma+1)\varphi^0_{xx} - l^2 - 2k\omega + \varepsilon\omega^2 = 0 \tag{2.3}$$

which has two roots (since $\varepsilon \ll 1$)

$$\omega_1 = 2\frac{k}{\varepsilon} + \frac{k}{2}\Omega, \quad \omega_2 = -\frac{k}{2}\Omega, \quad \Omega = K^{*0} + \frac{i}{k}(\gamma+1)\varphi^0_{xx} + \left(\frac{l}{k}\right)^2$$

If $\varphi^0_{xx} = u^0_x > 0$ (unperturbed flow is accelerated) the root ω_1 has a positive imaginary part and small perturbations increase, implying instability of the flow. In the case when $\varphi^0_{xx} = u^0_x < 0$ (unperturbed flow decelerates) perturbations corresponding to the second root ω_2 will increase, and again the flow is unstable.

As $\varepsilon \to 0$ the root ω_1 drops out and we obtain the usual result of gas dynamics [7, 11, 14] on the instability of transonic decelerated flow. However, it is clear that the result concerning the stability of accelerated transonic flow is largely associated with the use of the transonic approximation and neglect of the root ω_1.

For steady accelerated transonic flows with steady perturbations, it remains true that the flow is shock-free [9, 10].

Perturbations corresponding to the root ω_1 propagate in the direction of the main (unperturbed) gas flow and do not, under normal conditions, have time to develop sufficiently strongly. However, if physico-chemical processes involving the energy release take place in the main flow, weak perturbations can grow very rapidly and lead to a change of flow pattern (with the occurrence of jumps or even the development of unsteady flow). These issues are analysed in a quasi-one-dimensional approximation in [9].

Notice that Eq. (2.3) does not contain derivatives with respect to the y coordinate, and hence the change in the flow parameters along the y axis (across the direction of the main flow) is only taken into account indirectly (φ^0_{xx} can depend on y). It follows from (2.3) that the flow is neutrally stable with respect to perturbations propagating along the y axis (with $k = 0$). Thus, the analysis of the stability of transonic flow using the modified equation (1.7) gives incomplete and incorrect information. These results will be refined below in Section 5.

3. THE FORMATION OF UNSTEADY SHOCK WAVES IN TRANSONIC FLOWS

The problem of the development of shock waves in transonic flows is far from being finally solved. The importance of creating flow without shocks in applications, which encouraged the initial interest in the development and existence of shocks in flows of this kind, has led to the formulation of one of the most fundamental problems of gas dynamics.

We know of no investigations of the development of unsteady shock waves in transonic flows.

We will consider the development over time of the envelope of characteristics (1.8), denoting the formation of a shock wave. We choose two centres—the perturbation source, with coordinates (x_{01}, y_0), (x_{02}, y_0) (and identical coordinate) (Fig. 2). Using Eq. (1.8) we compute $\partial x/\partial x_0$ along the ray $y = y_0$ (where $\partial y/\partial x_0 = 0$, $\partial y_0/\partial x_0 = 0$). We have

$$\frac{\partial x}{\partial x_0} = 1 + \frac{\gamma+1}{2}\frac{t^2}{t - \varepsilon(x - x_0)}\frac{\partial u}{\partial x_0} = 0 \tag{3.1}$$

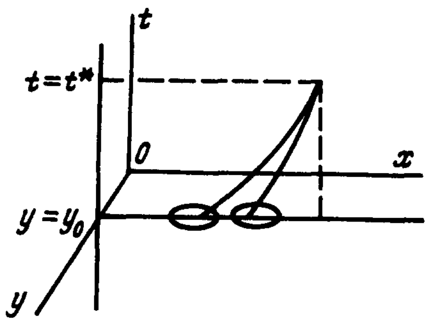

Fig. 2.

from which it follows when $\varepsilon = 0$ that

$$\frac{\partial u}{\partial x_0} = -\frac{2}{\gamma+1}\frac{1}{t} < 0 \tag{3.2}$$

A negative velocity gradient means that the envelope arises only for compression perturbations, and the earlier it is (the smaller t is), the larger the absolute magnitude of the velocity gradient will be.

Relation (3.1) implies that when $\varepsilon \neq 0$ the envelope only occurs for compression perturbations, the envelope arising faster (what the same initial amplitude of perturbation) in downstream ($x > x_0$) perturbations. However, an analysis based on the LRT equation does not allow entirely for perturbations propagating downstream.

We will consider the parts of wave fronts which move along the ray $x = x_0$. We will compute $\partial y/\partial y_0$ (in this case $\partial x/\partial y_0 = 0$, $\partial x_0/\partial y_0 = 0$). We have

$$\frac{\partial y}{\partial y_0} = 1 - \frac{\gamma+1}{2}\frac{t^2 - \varepsilon(y-y_0)^2}{(y-y_0)(1+\varepsilon K^*)}\frac{\partial u}{\partial y_0} \tag{3.3}$$

For those parts of the wave fronts which are near-flat, $\partial u/\partial y_0 = 0$ and reversal does not occur (according to (3.3) $\partial y/\partial y_0 = 1$).

When $\varepsilon = 0$ Eq. (3.3) gives the condition under which the characteristics intersect, whence it follows that

$$\frac{\partial u}{\partial y_0} = \frac{2}{\gamma+1}\frac{y-y_0}{t^2} \tag{3.4}$$

Using the condition for potential flow, we find $\partial v/\partial y_0$ from (3.4). After differentiating with respect to y_0 and integrating with respect to x_0 we have

$$\frac{\partial v}{\partial y_0} = -\frac{2}{\gamma+1}\frac{x_0}{t^2} + V(y, y_0)$$

The first term is a negative quantity which describes the profile of the compression wave, the second is an undefined function, and full information about reversal of the wave front cannot be obtained. Hence, incomplete or incorrect results are obtained for those parts of the wave fronts which propagate across the direction of the main flow when the modified Eq. (1.7) is used to explore the formation of shock waves in transonic flow.

4. THE ENERGY INTEGRALS FOR THE EQUATIONS OF UNSTEADY TRANSONIC FLOW

In the theory of partial differential equations, estimates for the solution are obtained from the energy integral; the use of this integral means that a uniqueness theorem can be proved for the solutions of these equations. A detailed analysis of these problems is made in [18, 19], for example. The modified equation introduced in Section 1 can be used similarly to analyse unsteady transonic flow. Unfortunately, it is not perfect (see the analysis of mixed problems below).

We differentiate Eq. (1.7) with respect to x, introduce a new unknown function $u = \varphi_x$ and after multiplying by u_t write the equation in divergence form

$$-\left(\frac{K^*}{2}(u_x)^2\right)_t+(K^*u_xu_t)_x-\left(\frac{1}{2}(u_y)^2\right)_t+(u_yu_t)_y-((u_t)^2)_x-\left(\frac{\varepsilon}{2}(u_t)^2\right)_t=Q \tag{4.1}$$

$$Q=-\frac{1}{2}K_t^*(u_x)^2$$

We now integrate Eq. (4.1) over the region V, which is the interior of the dependence cone, the base of which is the plane $t = 0$. The surface S of that region is formed by surfaces of the dependence cone S_l, and S_0 is the part of the plane $t = 0$ cut out by that cone (Fig. 3). Applying the Gauss–Ostrogradskii theorem, we obtain

$$\iint_S q\,dS=\iiint_V Q\,dV \tag{4.2}$$

$$q=\tau\varkappa+((K^*u_x-u_t)u_t)\xi+(u_yu_t)\eta,\quad \varkappa=-\frac{1}{2}(\varepsilon(u_t)^2+(u_y)^2+K^*(u_x)^2) \tag{4.3}$$

(τ, ξ and η are the components of the unit vector of the outward normal to S, along the t, x and y axes).

We make one more section (S_1) of the characteristic cone by the plane $t = t_1 > 0$. Then as in the case of (4.2) we have

$$\iint_{S_l} q\,dS+\iint_{S_1} q\,dS+\iint_{S_0} q\,dS=\iiint_V Q\,dV \tag{4.4}$$

(V is the interior of the dependence cone with base $t = 0$ truncated by the plane $t = t_1$).

By the definition of the characteristic cone, the quadratic form (4.3) is non-negative on its surface. Then it follows from (4.4) that

$$\iint_{S_1} \varkappa\,dS\leqslant\iint_{S_0}\varkappa\,dS+\iiint_V Q\,dV \tag{4.5}$$

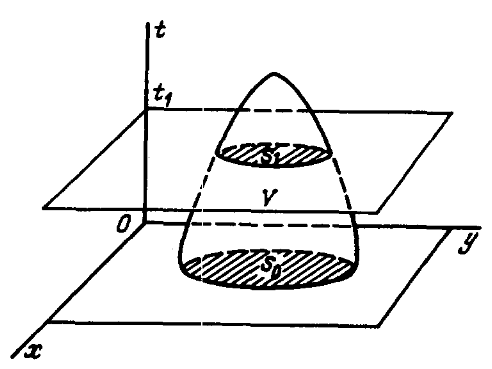

Fig. 3.

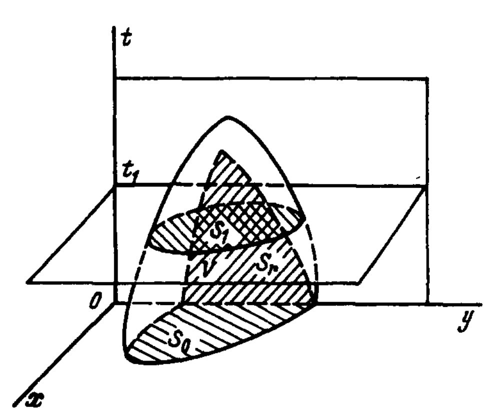

Fig. 4.

(on S_0 the unit vector of the outward normal is (–1, 0, 0), on S_1–(1, 0, 0)). In (4.5) the integrand Q can be evaluated in terms of the integrands of surface integrals, and by changing from a volume to a repeated integral, limits for the solution of the equation can be obtained.

We introduce the notation

$$I(t_1) = \iint_{S_1} \varkappa dS$$

so that (4.5) can be rewritten in the form of the integral inequality

$$I(t_1) \leqslant I(t_0) + M\int_{t_0}^{t_1} I(\hat{t})d\hat{t}$$

or the differential inequality $dI/dt \leqslant MI$ with solution $I \leqslant I(t_0)e^{Mt}$.

It is obvious that if $I(t_0) = 0$, then $I \equiv 0$ for any $t_1 > t_0$ inside the cone of dependence of the solution on the initial data. Thus, since the initial conditions are identically zero, the solution remains zero at all subsequent times. Hence, any non-trivial solution inside the cone of dependence is unique.

Note that a similar proof cannot be given for solutions of the LRT equation because that equation has no dependence cone. The non-trivial characteristics which leave the spatial surface t = const form a paraboloid of influence. The Cauchy problem for the LRT equation has either no solution at all or an infinite number of solutions.

Aspects of the correct formulation of a number of mixed problems of gas dynamics have been considered in [20]. The mixed problem turns out to be well-posed (it has a solution which is unique and has a continuous dependence on the initial and boundary data) inside the dependence cone if the energy integral is dissipative. A necessary condition for this is that the boundary conditions should be dissipative.

For mixed problems, the boundary of the region V in which the energy integral is constructed (Fig. 4) consists of part of the surface of the characteristic cone, sections of that cone by surfaces $t = 0$, $t = T$, and also S_Γ, sections of the cone by the surface $x = 0$ (or another plane $Ax + By$ = const), where the boundary conditions are assigned.

Integrating relation (4.1) over the region V and applying the Gauss–Ostrogradskii theorem, we obtain an inequality which differs from (4.5) in having the additional term

$$-\iint_{S_\Gamma} (K^* u_x - u_t)u_t dy dt$$

on the left-hand side (the unit outward normal to S_Γ is (0, 0, –1).

By definition [20], the boundary conditions are dissipative if

$$u_t(K^* u_x - u_t)|_{x=0} \leqslant 0 \tag{4.6}$$

In this case, the energy integral is obtained by the same technique as in the Cauchy problem.

The physical meaning of condition (4.6) is as follows. For supersonic flow ($K^* < 0$) which accelerates both in the direction of the x axis ($u_x > 0$) and over time ($u_t > 0$), condition (4.6) is satisfied and the boundary conditions are dissipative. In that case inequality (4.5) is satisfied, and thus the Cauchy problem for transonic accelerated flows is well-posed, that is consistent with the results of Section 2 concerning the stability of such accelerated flow.

If the velocity increases with time ($u_t > 0$), but ($u_x < 0$) (considering the region of deceleration of supersonic flow $K^*u_x > 0$), and if also $K^*u_x > u_t$, then condition (4.6) is not satisfied and the Cauchy problem is ill-posed. In the case when $K^*u_x < u_t$ the opposite is true. Thus, in supersonic unsteadily accelerated flow, small perturbations have the capacity to grow and cause the development of instability only in regions of strong spatial deceleration of the flow.

If supersonic flow is decelerated unsteadily, instability can develop at quite a large positive gradient u_x.

In zones of subsonic flow ($K^* > 0$) during unsteady acceleration ($u_t > 0$), one can expect the perturbations also to grow in regions of flow acceleration (at sufficiently large gradients u_x, so that $K^*u_x > u_t$). We then have completely the opposite picture to that of supersonic flow.

Note that very different results are obtained if the boundary conditions are assigned in the $y = 0$ plane instead of the $x = 0$ plane. The inequality then obtained differs from (4.5) in having on the left-hand side the additional term

$$-\iint_{S_\Gamma} u_y u_t \, dx dt$$

(the unit outward normal to S_Γ is now (0, 0, −1).

The boundary conditions are dissipative if

$$u_y u_t \,|_{y=0} \leqslant 0 \tag{4.7}$$

Condition (4.7) is satisfied, for instance, when $u_t > 0$, $u_y < 0$. For such flows at each instant of time the longitudinal component of the flow velocity decreases with distance from the axis of symmetry of the flow $y = 0$. Condition (4.7) is obviously not satisfied for unsteady Taylor flow, realized during the start-up (acceleration) of a transonic nozzle. For unsteadily decelerated transonic flow we have $u_t < 0$ and (4.7) is satisfied when $u_y > 0$, that is, for flows with a longitudinal velocity component which increases with distance from the axis of symmetry of the flow $y = 0$ (like Taylor flows which occur during deceleration of a transonic nozzle, for instance).

Comparing (4.6) and (4.7), we can conclude that the proof of the stability of transonic flow based on the energy integral for the modified Eq. (1.7) depends very much on the position in space of the line on which the boundary conditions are assigned and therefore the results apply only to a limited extent, and are sometimes incorrect.

5. ALLOWING FOR HIGHER APPROXIMATIONS OF THE TRANSONIC EXPANSION

Generalizing the above with respect to modification of the equation of unsteady transonic flow, and taking the defects of the equations that have been mentioned into account, it seems reasonable to use an equation which includes terms of the next order of smallness (order $\delta^2 = \varepsilon$) to analyse transonic flows. From (1.4) we have (the small constant terms εK and εK^2 are omitted)

$$\begin{aligned} &K^*\varphi_{xx} + \varphi_{yy} - 2\varphi_{xt} - \varepsilon\left[\left((\gamma-1)\varphi_t + \frac{\gamma+1}{2}(\varphi_x)^2\right)\varphi_{xx} + 2\varphi_y\varphi_{xy} + \right. \\ &\left. +(\gamma-1)\varphi_x\varphi_{yy} + 2\varphi_x\varphi_{xt} + \varphi_{tt}\right] = 0 \end{aligned} \tag{5.1}$$

The term φ_{tt} corresponds to singular perturbations. There are no other singular terms, but their derivatives and combinations of such φ_t, $(\varphi_x)^2$, φ_y, φ_{xy} (for example, when $(\varphi_x)^2 \sim \varepsilon^{-1}$ the quantity φ_x is not very large, but allowance must be made for $(\varphi_x)^2$, and other combinations of derivatives occurring in (5.1) might be large. Moreover, the remaining non-linear terms of order ε have a different physical meaning: $\varphi_y\varphi_{xy}$ and $\varphi_x\varphi_{yy}$ describe the non-linearity in a transverse direction to basic transonic flow; $\varphi_t\varphi_{xx}$ and $\varphi_x\varphi_{xt}$ have an additional non-linearity, which appears during unsteady motion; $(\varphi_x)^2\,\varphi_{xx}$ has an additional non-linearity in the direction of the main flow.

The characteristic surfaces defined by Eq. (5.1) satisfy the relation (where terms of order known to be higher than ε are omitted)

$$\begin{aligned} &K_1^* t^2 - K_3 t(x-x_0) - K_4 t(y-y_0) - \varepsilon(x-x_0)^2 = \left(\frac{K_3}{4K_2} + \varepsilon K^*\right)(y-y_0)^2 \\ &K_1^* = K^* - \varepsilon\left((\gamma-1)\varphi_t + \frac{\gamma+1}{2}(\varphi_x)^2\right), \quad K_2 = 1 - \varepsilon(\gamma-1)\varphi_t, \\ &K_3 = -2(1+\varepsilon\varphi_x), \quad K_4 = -2\varepsilon\varphi_y \end{aligned} \tag{5.2}$$

For constant coefficients K_1^*, K_2, K_3, K_4, K^*, for each fixed t Eq. (5.2) gives an ellipse in the plane of the variables x, y. The ellipse becomes larger in time, its foci and centre moving along the x axis. Generally speaking, the coefficients in Eq. (5.2) are not constant and the signs might change in different regions of the field of flow.

The equation of the envelope of characteristics (5.2) over time is governed by the following equation (the coefficients are assumed constant and the expansion in terms of ε is taken into account)

$$(x-x_0)=\left\{\frac{K_4}{2}\pm\sqrt{-K^*}\left[1-\frac{\varepsilon}{2K^*}\left(\frac{\gamma+1}{2}(\varphi_x)^2+(\gamma-1)(\varphi_t+\varphi_x)\right)\right]\right\}(y-y_0) \tag{5.3}$$

We see (as in (1.10)) that the envelope exists only in supersonic flow $K^* < 0$. The term $K_4/2$ in (5.3) corresponds to propagation of the disturbance in a direction transverse to the main flow at a velocity $2\varepsilon\upsilon$. This velocity can be assumed to be approximately constant and to have a different direction on different sides of the $y = y_0$ axis. Thus, the envelope of the wave fronts from a point source of perturbations is steeper than (1.10). Moreover, the slope increases, on account of the term in (5.3) associated with $(\varphi_x)^2$, and can decrease or increase on account of the terms containing φ_t and φ_x, taken with the appropriate sign.

To investigate the stability of transonic flow in relation to weak (linear) unsteady perturbations, as usual we linearize this equation and, taking its coefficients to be constant, look for a solution proportional to $\exp(-i\omega t + ikx + ily)$. We obtain a dispersion relation which (using the smallness of ε) has roots

$$\omega_{1,2}=\frac{k}{\varepsilon}\Big[1-\frac{\varepsilon}{2k}(i(\gamma-1)\varphi^0_{xx}-2k\varphi^0_x)\pm(1-\frac{\varepsilon}{2k}(i(\gamma-1)\varphi^0_{xx}-2k\varphi^0_x)-$$
$$-\frac{\varepsilon}{2k^2}\Big\{-k^2K^{*0}-ik(\gamma+1)\varphi^0_{xx}-l^2-\varepsilon(\gamma+1)ik\varphi^0_x\varphi^0_{xx}+\varepsilon k^2\left((\gamma-1)\varphi^0_t+\frac{\gamma+1}{2}(\varphi^0_x)^2\right)-$$
$$-2il\varepsilon\varphi^0_{xy}+2\varepsilon kl\varphi^0_y+(\gamma-1)\varepsilon ik\varphi^0_{yy}+(\gamma-1)\varepsilon l^2\varphi^0_x-2\varepsilon ik\varphi^0_{xt}\Big\})\Big] \tag{5.4}$$

From the form of the roots (5.4) it can be concluded that any inhomogeneity of the flow expressed by the derivatives $\varphi^0_{yy} = \upsilon_y$, $\varphi^0_{xy} = \upsilon_x = u_y$, $\varphi^0_{xt} = u_t$ and functions $\varphi^0_y = \upsilon$, $\varphi^0_x = u$ leads to a growth of perturbations corresponding to either the first or the second root of (5.4).

On the basis of Eq. (5.1), we consider the development of unsteady shock waves from wave fronts which propagate across the direction of the main flow from two sources on the $x = x_0$ axis with ordinates y_{01}, y_{02}. Using Eq. (5.2) we calculate $\partial y/\partial y_0$ along the ray $x = x_0$, and have

$$\frac{\partial y}{\partial y_0}=1+\frac{t}{2(y-y_0)}\left[(y+1)t\frac{\partial u}{\partial y_0}\left(1+\varepsilon\left((\gamma+3)u-K-(\gamma-1)\varphi_t+\frac{vt}{y-y_0}\right)\right)+\right.$$
$$\left.+\varepsilon t(\gamma-1)\frac{\partial\varphi_t}{\partial y_0}+\varepsilon t(\gamma+1)u\frac{\partial u}{\partial y_0}-2\varepsilon\frac{\partial v}{\partial y_0}(y-y_0)\right]$$

For near-planar fronts $\partial u/\partial y_0 = 0$ and the last term will have the main influence, that is

$$\frac{\partial y}{\partial y_0}=1+\varepsilon\frac{\partial v}{\partial y_0}t$$

For a shock wave to develop, we must have

$$\frac{\partial v}{\partial y_0}=-\frac{1}{\varepsilon t}<0$$

which corresponds to the result (3.3) (the wave fronts of compression perturbations are reversed).

The equation of unsteady transonic flow in modified form (5.1) is also invariant under an interchange of the x, y axes. Thus, the condition for dissipative boundary conditions in the mixed problem will, as before, largely depend on the choice of the line on which the boundary conditions are assigned.

We can, however, determine what allowing for additional terms of the transonic expansion will achieve. We will consider the energy integral for the mixed problem on the basis of Eq. (5.1).

The requirement that the boundary conditions shall be dissipative is now:

when the boundary conditions are given on the $x = 0$ axis

$$u_t\left[K^*u_x - u_t - \varepsilon\left((y-1)\varphi_t + \frac{\gamma+1}{2}u^2\right)u_x + \varphi_y u_y + uu_t\right]\Bigg|_{x=0} \leqslant 0 \tag{5.5}$$

when the boundary conditions are given on the $y = 0$ axis

$$u_t[u_y - \varepsilon(\varphi_y u_x + (\gamma-1)uu_y)]|_{y=0} \leqslant 0 \tag{5.6}$$

Inequalities (5.5) and (5.6) are obviously very different.

If the quantities $\Phi_y = \upsilon$, u_y are large, instead of (5.6) we can use the inequality

$$u_t[u_y - \varepsilon \upsilon u_x]|_{y=0} \leqslant 0$$

This inequality is satisfied, for example, when $u_t > 0, u_y > 0, \upsilon > 0, u_x > 0, u_y < \varepsilon\upsilon u_x$ (which is true for unsteady Taylor flows realized during start-up of a transonic nozzle).

Thus, if Eq. (5.1) is used instead of (1.7), a result can be obtained on the stability of unsteady accelerated Taylor flow for sufficiently large υ, u_x.

We will now consider the remaining terms in (1.4) of higher order of smallness with respect to δ. For the third and higher approximations, it is no longer possible to assume that the perturbed flow is isentropic. The curved weak shock waves that arise generate entropy gradients (the entropy jump at these discontinuities is a third-order quantity with respect to the amplitude of the pressure jump) which, according to Crocco's formula, determine the flow vorticity. It can be assumed that the velocity components of a gas in perturbed flow are the sum of a potential and small non-potential component. The derivatives of the latter occur in the third approximation, in accordance with formula (1.5).

We now consider the remaining terms of the third order of smallness (order ε^2), defined by the potential in Eq. (1.4)

$$\frac{\gamma-1}{2}\varphi_y^2\varphi_{xx} - 2\varphi_x\varphi_y\varphi_{xy} - (\gamma-1)\varphi_t\varphi_{yy} - 2\varphi_y\varphi_{yt} \tag{5.7}$$

The only new derivative not allowed for in the previous approximations in (5.7) is φ_{yt}.

The only potential term of the fourth order of smallness (order ε^3) in (1.4)

$$-\frac{\gamma+1}{2}\varphi_y^2\varphi_{yy} \tag{5.8}$$

It contains no derivatives that have not already been taken into account.

As one might expect, the third and fourth approximations contain mainly small additions to the terms that have been allowed for in the previous approximations, showing that it is not essential to take into account higher (third and fourth) approximations of the transonic expansion when investigating transonic flows. Equation (5.1) appears to be quite suitable for analysing unsteady transonic flows. We merely note that even allowing for (5.7) and (5.8), the equation is not invariant under an interchange of the space axes. Thus the results obtained from the energy inequalities for mixed problems will depend on the choice of the line on which the boundary conditions are given.

6. CONCLUSION

When studying unsteady transonic flows, the singular equation considered here must obviously be used instead of the LRT. It gives a description of the flow pattern near the sound barrier which is better from a physical standpoint. It enables the behaviour in low- and high-frequency unsteady perturbations of the flow to be analysed, has no degenerate characteristic surfaces and allows for the construction of the cone of dependence of the solution on the initial data. The Cauchy problem for this equation is well-posed and, at least in the small, its solution is unique. It is necessary to use modifications of the equation of unsteady transonic flow which include higher approximations of the transonic expansion to investigate the features of transonic flows in a spatial (two-dimensional) formulation.

I became interested in the question of transonic flow largely as a result of reading the monograph by O. S. Ryzhov [2] and having numerous discussions with V. N. Diyesperov, to whom I offer my deep gratitude.

REFERENCES

1. LIN, C. C., REISSNER, E. and TSIEN, H. S., On two-dimensional non-steady motion of a slender body in a compressible fluid. *J. Mathematics and Physics*, 1948, **27**, 220–231.
2. RYZHOV, O. S., *The Investigation of Transonic Flows in Laval Nozzles*. Computing Centre, Academy of Sciences of the USSR, Moscow, 1965.
3. COLE, J. D. and COOK, L. P., *Transonic Aerodynamics*. North-Holland, Amsterdam, 1986.
4. SEVOST'YANOV, G. D., The equation of unsteady transonic flows of an ideal gas. *Izv. Akad. Nauk SSSR. MZhG*, 1977, **1**, 105–109.
5. EHLERS, F. E., A finite difference method for the solution of the transonic flow around harmonically oscillating wings. AIAA Paper No. 543, 1974.
6. SEVOST'YANOV, G. D., Some solutions of transonic equations. In *Aerodynamics*, ed. S. V. Fal'kovich. Izd. Sarat. Univ., Saratov, 1975, **4**(7), 47–55.
7. KYO, Y. H. and SEARS, W. R., Plane subsonic and transonic potential flows. In *General Theory of High-speed Aerodynamics*, ed. W. R. Sears. Princeton University Press, Princeton, NJ, 1954, pp. 490–582.
8. VAN DYKE, M., *An Album of Fluid Motion*. Parabolic Press, Stanford, CA, 1982.
9. BOGDANOV, A. N. and KULIKOVSKII, V. A., Steady weakly-perturbed transonic flows of an oscillating-relaxing gas. *Zh. Prikl. Mekh. Tekh. Fiz.*, 1993, **2**, 48–58.
10. BOGDANOV, A. N. and KULIKOVSKII, V. A., The development of shock waves in transonic flows of an oscillating-relaxing gas. *Mat. Modelirovaniye*, 1992, **4**(12), 155–159.
11. KANTROWITZ, A., Formation and stability of normal shock waves in steady channel flows. In *Fundamentals of Gas Dynamics*, ed. H. W. Emmons. Oxford University Press, London, 1958.
12. KULIKOVSKII, A. G. and SLOBODKINA, F. A., Equilibrium of arbitrary steady flows at the transonic points. *Prikl. Mat. Mekh.*, 1967, **31**, 593–602.
13. PRASAD, P., Nonlinear wave propagation on an arbitrary steady transonic flow. *J. Fluid Mech*, 1973, **57**(4), 721–737.
14. KUZ'MIN, A. G., *Nonclassical Equations of Mixed Type and their Application to Gas Dynamics*. Izd. Leningr. Gos. Univ., Leningrad, 1990.
15. BOGDANOV, A. N., Modelling of the transition regime of the operation of a transonic nozzle. *Mat. Modelirovaniye*, 1995, **7**(9), 1178–126.
16. GURUSWAMI, G. P. and GOORJIAN, P. M., Unsteady transonic simulation on a full-span–wing–body configuration. *AIAA Jnl*, 1988, **26**(12), 1450–1456.
17. KUO, H., On the stability of two-dimensional smooth transonic flows. *J. Aeronaut. Sci.*, 1951, **18**(1), 1–6.
18. COURANT, R., *Partial Differential Equations*. Interscience, New York, 1962.
19. GODUNOV, S. K., *The Equations of Mathematical Physics*. Nauka, Moscow, 1979.
20. BLOKHIN, A. M., *Energy Integrals and their Applications to Problems of Gas Dynamics*. Nauka, Novosibirsk, 1986.

Translated by R.L.